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Embedding an Arbitrary Function into a Tchebycheff Space

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In this paper it is determined precisely when a given function belongs to some Tchebycheff system.

1. INTRODUCTION

Let $\mathscr{F}(T)$ be the set of real-valued functions on any subset T of the real line. Let $U \subset \mathscr{F}(T)$ be any $(k+1)$-dimensional vector space over $\mathbb{R}$ (the real line). U is a *Tchebycheff space* (*T-space*) *of degree* k iff for every nonzero $u \in U$ the number of distinct zeros of u, $Z(u)$, and the number of alternations in sign of $u(t)$ with increasing t, $S^-(u)$, each do not exceed k. Any basis of a T-space is a T-system, as classically defined in terms of the permanence of sign and nonvanishing of the Haar determinant, and if $\{u_i\}_0^k$ is a T-system, then the linear space generated by these functions forms a T-space. The T-space U is a *Markov space* if there exists a chain of T-spaces U_i of respective degrees i, $i = 0, 1,\ldots, k-1$ such that $U_0 \subset U_1 \subset \cdots \subset U_{k-1} \subset U$.

A set of elements $t_i \in T$, $i = 0, 1,\ldots, m$, is said be a *weak alternation sequence* of length m for u if $t_{i-1} < t_i$ for $i = 1,\ldots, m$, and $(-1)^{i+j} u(t_i)\, u(t_j) \geq 0$; $i, j = 0, 1,\ldots, m$. Define $S^+(u)$ to be the supremum over all m such that there is a *weak* alternation sequence of length m for u. We show that $u \in \mathscr{F}(T)$ can be embedded into a T-space or equivalently, a Markov space iff $S^+(u) < +\infty$. In particular if $S^+(u) = k$, we construct a chain of T-spaces U_i of respective degrees i, $i = 0, 1,\ldots$ such that $u \in U_k$ and $U_0 \subset U_1 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots$.

Any function I such as S^+ defined on $\mathscr{F}(T)$ and taking values in the set of nonnegative integers and $+\infty$ is called an *indicator function* if for every $(k+1)$-dimensional subspace U, U is a T-space iff for all nonzero $u \in U$, $I(u) \leq k$. We show that S^+ is an indicator function which majorizes all the indicator functions. Also, there is no minimal indicator function. On the

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other hand, while Z is not in general an indicator function (it is for the class of continuous functions on an interval), $Z(u) \leqslant I(u)$ for all indicator functions I subject to a nominal normalizing condition.

Let $u_0, ..., u_k$ be any real-valued functions defined on an arbitrary set T of cardinality greater than k. For $t_0, ..., t_k \in T$ let $\mathbf{t} = (t_0, ..., t_k)$ and define the $(k+1) \times (k+1)$ matrix $V(\mathbf{t})$ (the *Haar matrix*) by $V_{ij}(\mathbf{t}) = u_j(t_i)$. Classically, a Tchebycheff system (or T-system) referred to a set $\{u_i\}_{i=0}^k$ of continuous functions defined on some closed real interval $[a, b]$ such that

$$\det V(\mathbf{s})\, V(\mathbf{t}) > 0, \tag{T}$$

whenever $s_0 < s_1 < \cdots < s_k$, $t_0 < t_1 < \cdots < t_k$. In this case of continuous functions on an interval, the condition (T) is equivalent to the *Haar condition*:

$$\det V(\mathbf{t}) \neq 0, \qquad \text{whenever } t_0, ..., t_k \text{ are distinct.}$$

This in turn is equivalent to the condition that the u_i's be linearly independent and that every nontrivial linear combination of them have at most k zeros (in $[a, b]$).

When each subset $\{u_i\}_{i=0}^n$, $n = 0, 1, ..., k$ is a T-system, $\{u_i\}_{i=0}^k$ has been called a *Markov system*, and when a T-system $\{u_i\}_{i=0}^k$ can be extended to a larger T-system $\{u_i\}_{i=0}^{k+1}$, the former T-system has been called *extendable*. If for every choice of points $s_0 < \cdots < s_k$, $t_0 < \cdots < t_k$, $\det V(\mathbf{s})\, V(\mathbf{t}) \geqslant 0$, $\{u_i\}_{i=0}^k$ has been called a *weak T-system*.

Unfortunately, there has been no general agreement in the literature about which term to apply to which concept and the reader is cautioned accordingly. The terminology used in this paper has been chosen to reflect both historical precedent as closely as possible and also the functional requirement that the important classifications be named suggestively and succinctly.

Extensive studies of such T-systems and Markov systems can be found [1; 3; 4; 6] and others. T-systems of continuous functions on open or half-open intervals (see [3]) or on compact sets (see [1; 3, Chap. VII; 5]) have also been dealt with. Work with T-systems of arbitrary real-valued functions defined on an arbitrary partially ordered set appears to have surfaced first in Rutman [7], and is further developed in Zielke [8; 9]. Combining results from the last two papers gives

THEOREM (Zielke). *Any T-space on a dense subset of an open interval is an extendable Markov space.*

In his earlier paper [8], Zielke provides the example 1, $t \sin t$, $t \cos t$ on $[0, \pi]$ of a T-space of infinitely differentiable functions on a closed interval which is not a Markov space. In fact, for every dimension and for half-open as well as closed intervals, there are examples of non-Markov T-spaces in Zielke [10].

Fundamental to both of Zielke's papers is his investigation of alternation properties which characterize T-spaces. While there is in the literature some prior mention of such properties (e.g. [2; 3, Chap. VII; 5, Sect. 3, paragraph 2]) they typically have been overlooked in the study of T-spaces, most probably because for T-spaces of continuous functions on an interval they are trivial. However, for T-spaces of arbitrary functions on arbitrary domains they are essential.

For, while a set of continuous functions on an interval is a T-system if and only if the functions satisfy the Haar condition, and while in fact a $(k+1)$-dimensional linear space of such functions on an open interval is an extendable Markov space if and only if each nonzero function has no more than k zeros, for arbitrary functions on an arbitrary domain these conditions are far apart. Specifically, if $\{u_i\}_{i=0}^{k}$ satisfies the Haar condition, we define its linear span U to be a *Haar space*. This is equivalent to the condition that $Z(u) \leqslant k$ for each nonzero $u \in U$ and the dimension of U is $k+1$. If in addition the domain of the elements of U is linearly ordered and $S^-(u) \leqslant k$ for each $u \in U$ then it follows that $\{u_i\}_{i=0}^{k}$ is a T-system as defined by condition (T), and conversely, in which case U is a T-space. (Equivalently, if the dimension of U is $k+1$ then U is a T-space if and only if $S^-(u) \leqslant k$ for each nonzero $u \in U$.)

Clearly, extendable Markov space $\Rightarrow$ Markov space $\Rightarrow$ T-space $\Rightarrow$ Haar space, and as we have said, when the space consists of continuous functions on an open interval, all these implications are reversible. When the functions are arbitrary, even on an open interval, the last implication is not in general reversible, but by Zielke's theorem the first two nonetheless are. The previously mentioned example shows that the second implication is not reversible in general.

If $a, b \in \mathbb{R}$, $a < b$, then $[a, b]$, $]a, b[$ and $[a, b[$ (or $]a, b]$), are, respectively, the closed, open, and half-opened intervals between a and b. For any set T, card T denotes the cardinality of T. If $T \subset \mathbb{R}$, the closure of T (in $\mathbb{R}$) is denoted cl T. The letter V is reserved to denote the Haar matrix (with respect to inferred $u_0, \ldots, u_k$). The next lemma follows from linear algebra.

(1.1) Lemma. *For any set T, the set $\{u_i\}_{i=0}^{k} \subset \mathscr{F}(T)$ is linearly independent if and only if there are distinct $t_i \in T$ $(0 \leqslant i \leqslant k)$ such that* det $V(\mathbf{t}) \neq 0$.

It follows that if $u_0, \ldots, u_k$ are linearly independent, then card $T \geqslant k+1$.

(1.2) Corollary. *If $\{u_i\}_{i=0}^{k}$ is a linearly independent subset of $\mathscr{F}(T)$ then for some $t_0, \ldots, t_k \in T$ and any $a_0, \ldots, a_k \in \mathbb{R}$ there is a u in the linear space spanned by $\{u_i\}_{i=0}^{k}$ satisfying $u(t_i) = a_i$ $(0 \leqslant i \leqslant k)$.*

2. Definitions, Basics

Let T be an arbitrary set and suppose $\{u_i\}_{i=0}^k \subset \mathscr{F}(T)$. Then $\{u_i\}_{i=0}^k$ is called a *Haar system* of *degree* k if the Haar condition (see above) is satisfied. If T is a linearly ordered set then $\{u_i\}_{i=0}^k$ is called a *T-system* of *degree* k if the condition (T) is satisfied. When furthermore $\{u_i\}_{i=0}^n$ is a T-system of degree n for $n = 0, 1,..., k$ then $\{u_i\}_{i=0}^k$ is called a *Markov system* of *degree k*.

The linear span of a Haar (respectively, T-, Markov) system of degree k is called a *Haar space* (respectively, *T-space*, *Markov space*) of *degree k*. Notice that a Haar, T- or Markov space of degree k has dimension $k + 1$ and that any basis of a Haar (respectively, T-) space is a Haar (T-) system of the same degree. Furthermore, the restriction of any such space of degree k to a subset of cardinality at least $k + 1$ remains such a space of degree k. And in view of the defining Haar condition and condition (T), if $\{u_i\}_{i=0}^k$ is a Haar system on a set T (respectively, T- or Markov system on a linearly ordered set T) and $\theta: T \to T'$ is a 1–1 map to a set T' (respectively, a strictly increasing or strictly decreasing map to a linearly ordered set T'), then $\{u_i \circ \theta^{-1}\}_{i=0}^k$ is a Haar system (respectively, T- or Markov system) on $\theta(T)$.

The term "polynomial," sometimes appearing in the literature to denote an element of an arbitrary space, is here reserved exclusively to denote an algebraic polynomial (an element of the T-space with basis $1, t, t^2,..., t^k$).

The following easily proved lemma is needed in what follows.

(2.1) Lemma. *Any element u of a Haar space U of degree k having k (distinct) zeroes $t_1 ,..., t_k$ is a scalar multiple of a determinant function*: $u(t) = \alpha \det V(t, t_1 ,..., t_k)$ *for some* $\alpha \neq 0$. *If U is a T-space and* $t_1 < \cdots < t_k$, *then after possibly multiplying u by* -1, $t \in]t_i , t_{i+1}[\Rightarrow (-1)^i \phi(t) < 0$ *for* $i = 0,..., k$ (*with* $t_0 \equiv -\infty$, $t_{k+1} \equiv +\infty$).

3. Equivalent Characterizations of T-Spaces

Let U be a $(k + 1)$-dimensional subspace of $\mathscr{F}(T)$. In the preceding section we defined U to be a T-space in terms of a basis for U. However one can characterize a T-space by properties of the elements of U without explicit mention of a basis. If U is a $(k + 1)$-dimensional subspace of continuous functions on a closed interval $[a, b]$, it can be shown (see [3, p. 22]) that if every nontrivial element of U vanishes at no more than k points in $[a, b]$ then U is a T-space on $[a, b]$. However, for functions that are not continuous the number of points at which the functions are zero is not sufficient to characterize T-spaces. As a trivial example let $T = \mathbb{R}$ and $u_0(t) = -1$ if $t \leqslant 0$, $u_0(t) = 1$ if $t > 0$.

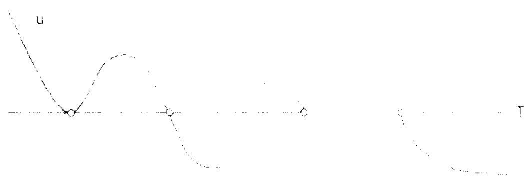
DEFINITION. Suppose T is a partially ordered set and $u \in \mathscr{F}(T)$. An *alternation sequence* of *length n* for u on T is a set $\{x_0, \ldots, x_n\} \subset T$ satisfying $x_0 < \cdots < x_n$ and such that $(-1)^{i+j} u(x_i) u(x_j) > 0$ $(0 \leqslant i, j \leqslant n)$. (This, of course, is equivalent to $u(x_i) u(x_{i+1}) < 0$ when $n > 0$.) The supremum of n taken over all alternation sequences of length n for u on T is denoted $S^-(u)$.

Now suppose $T_1, T_2, \ldots$ are pairwise disjoint subsets of T such that u never vanishes on T_i $(i = 1, 2, \ldots)$, and such that for each i, $a, b \in T_i$, $t \in T$, $a < t < b \Rightarrow t \in T_i$ (e.g., if $T \subset \mathbb{R}$ then each $T_i = T \cap I$ for some interval I). With $u|_{T_i}$ denoting the restriction of u to T_i, the supremum of $\sum_i S^-(u|_{T_i})$ taken over all such sets $\{T_1, T_2, \ldots\}$ is denoted $S^0(u)$.

The notation S^- and S^+ is consistent with that in [2-4]; while in these sources the definitions of S^- and S^+ are in terms of the number of sign changes in related sequences, our definitions here are equivalent to the others.

For any set T, the number of distinct elements $t \in T$ such that $u(t) = 0$ (the *zeros* of u in T) is denoted $Z(u)$.

A *double zero* t of $u \in \mathscr{F}(T)$ is a zero of u such that for some $r, s \in T$, $r < t < s$ and for any $x, y \in T$ satisfying $r \leqslant x < t < y \leqslant s$, $u(x)\, u(y) > 0$. The number of double zeros of u is denoted $D(u)$. Of course, in the case of ordinary polynomials, our "double zero" applies to any zero of even order (see Fig. 1).



$S^0(u) = 1$, $S^-(u) = 3$, $Z(u) = 4$, $D(u) = 1$, $S^+(u) = 7$

FIGURE 1

For our purposes, the domain of a function is unique and implicit in the definition of the function. Thus, the restrictions of a function to two different subsets of its domain are to be considered for notational purposes as two different functions. The restriction of a function u to a set S is denoted $u|_S$.

(3.1) LEMMA. *For any $u \in \mathscr{F}(T)$, $u \neq 0$ and T linearly ordered,* $S^0(u) \leqslant S^-(u) \leqslant \max\{S^-(u), Z(u)\} \leqslant S^0(u) + Z(u) \leqslant S^0(u) + Z(u) + D(u) \leqslant S^+(u)$.

Proof. Suppose $T_1, T_2, \ldots$ are pairwise disjoint subsets of T on which u never vanishes as above, ordered so that $\sup T_i < \inf T_{i+1}$. The concatenation of an alternation sequence of length n in T_i with one of length m in T_{i+1} will, after possibly excluding the first point in the second sequence, form an alternation sequence of length at least $n + m$ on $T_i \cup T_{i+1}$. Hence, $S^0(u) \leqslant S^-(u)$.

Conversely, an alternation sequence of length n on T can be partitioned into subsets P_i, $i = 1,..., m$ of maximal cardinality subject to the constraints that each P_i contains only elements which are consecutive in the original alternation sequence, and that in the convex hull of P_i (for each $i = 1,..., m$) u does not vanish. Then by definition $S^0(u) \geqslant \sum_i S^-(u_i) \geqslant n - m + 1$ where $u_i = u|_{P_i}$. In as much as between any two P_i's (ordered on T) there necessarily lies a zero of u (by the maximality condition), $S^-(u) \leqslant S^0(u) + Z(u)$. Hence the second and third inequalities also hold.

The fourth inequality is trivial.

For the last inequality, we can assume without loss of generality that $Z(u) < +\infty$. Suppose $T_1, T_2,...$ are as above. Let an alternation sequence of finite length be chosen in each T_i and let $x_0, x_1,...$ be the natural linear ordering of the set formed by all the respective alternation sequences (one for each T_i) and all the zeros of u. By discarding (if necessary) from $x_0, x_1,...$ the first element from any of the alternation sequences chosen in $T_2, T_3,...,$ respectively, the remaining points relabeled $y_0 < y_1 < y_2 < \cdots$ can be formed into a generalized alternation sequence for u. Since there are $n + 1$ points in an alternation sequence of length n, it follows that $S^0(u) + Z(u) \leqslant S^+(u)$. Furthermore, if y_i is a double zero, there exist $r, s \in T$ such that for all $x, y \in T$, $r \leqslant x < y_i < y \leqslant s$ implies $u(x)\,u(y) > 0$. In this case, either the generalized alternation sequence $y_0, y_1,...$ can be augmented by the inclusion of one or both of r or s, or else $y_m := \min\{y_n \mid n > i, u(y_n) \neq 0\}$ was the first element in the alternation sequence chosen from some T_j (thus undiscarded). In either case an extra point exists in the generalized alternation sequence on behalf of y_i. The nonzero elements of the possibly augmented sequence can be decomposed into new T_i's as above, and the previous argument repeated for each double zero. Hence $S^0(u) + Z(u) + D(u) \leqslant S^+(u)$.

(3.2) *Notes.* 1. If T is a real interval and u is continuous, $S^0(u) = 0$ whence $S^-(u) \leqslant Z(u)$.

2. If u is a polynomial, $S^+(u) \leqslant \deg u$.

3. If T is an open interval and u is a polynomial, $S^-(u)$ is exactly the number of zeros of u in T of odd index, and $S^+(u) = Z(u) + D(u)$.

4. All the inequalities in Lemma 3.1 can be simultaneously strict (see Fig. 1).

It was noted earlier that continuous functions defined on an interval T form a T-space of degree k if and only if the only element with more than k zeros is 0. This equivalence is true in general for a Haar space:

(3.3) LEMMA. *Suppose U is a $(k+1)$-dimensional linear subspace of $\mathscr{F}(T)$ (T arbitrary). Then U is a Haar space if and only if $Z(u) \leqslant k$ for every nonzero $u \in U$.*

While a Haar space is not in general a T-space, the following equivalences do obtain.

(3.4) THEOREM. *Let T be an arbitrary linearly ordered set and let U be a $(k+1)$-dimensional linear subspace of $\mathscr{F}(T)$. Then the follwoing are equivalent:*

(1) *U is a T-space of degree k;*

(2) *$S^-(u) \leqslant k$ and $Z(u) \leqslant k$ whenever $0 \neq u \in U$;*

(3) *$S^0(u) + Z(u) \leqslant k$ whenever $0 \neq u \in U$;*

(4) *$S^+(u) \leqslant k$ whenever $0 \neq u \in U$.*

Proof. (1) $\Rightarrow$ (2). As in [8], Lemma 2(a) $\Rightarrow$ (b).

(2) $\Rightarrow$ (4). As in [8] Lemma 2(b) $\Rightarrow$ (c).

(4) $\Rightarrow$ (3). This is a direct consequence of Lemma 3.1.

(3) $\Rightarrow$ (1). Let $\{u_i\}_{i=0}^k$ be any basis for U. By Lemma 1.1 there are elements of T, say $t_0 < \cdots < t_k$ such that with respect to $\{u_i\}_{i=0}^k$, $\det V(t_0, \ldots, t_k) \neq 0$, say $\det V(t_0, \ldots, t_k) > 0$.

Now suppose $s_0 < \cdots < s_k$ are any other elements of T. It suffices to show that $\det V(s_0, \ldots, s_k) > 0$. Define $r_i = \min\{s_i, t_i\}$ $(i = 0, \ldots, k)$. Then $r_i = \min\{s_i, t_i\} \leqslant s_i < s_{i+1}$ and similarly $r_i < t_{i+1}$ so $r_i < r_{i+1}$ $(i = 0, \ldots, k-1)$.

For $i = 0, \ldots, k$ define $\varphi_i(t) = \det V(r_0, \ldots, r_{i-1}, t, t_{i+1}, \ldots, t_k) \in U$. If $\varphi_i(t_i) > 0$ then $\varphi_i \neq 0$ whence by (3) φ_i has exactly k zeros (namely, $r_0, \ldots, r_{i-1}, t_{i+1}, \ldots, t_k$). It follows from (3) that in this case $S^0(\varphi_i) = 0$, whence $\varphi_i(t_i) > 0$ implies $\varphi_i(t) > 0$ whenever $t \in]r_{i-1}, t_{i+1}[\cap T$ $(i = 0, \ldots, k$; $r_{-1} = \inf T$, $t_{k+1} = \sup T)$.

Now $\varphi_0(t_0) > 0$ and $r_0 \in]r_{-1}, t_1[$ so $\varphi_0(r_0) > 0$. But $\varphi_0(r_0) = \varphi_1(t_1)$ so $\varphi_1(t_1) > 0$. Continuing in this fashion, we eventually obtain $\det V(r_0, \ldots, r_k) = \varphi_k(r_k) > 0$, that is, the sign of $\det V(t_0, \ldots, t_k)$ is the same as the sign of $\varphi_k(r_k)$.

Replacing t_i by s_i in the definition of φ_i, we analogously obtain that the sign of $\det V(s_0, \ldots, s_k)$ is the same as the sign of $\varphi_k(r_k)$, which has to be proved. (Note: This proof is similar to [8], Lemma 2(c) $\Rightarrow$ (a) where the author makes an unnecessary additional assumption.)

4. INDICATOR FUNCTIONS AND EMBEDDING

In Section 3 it was demonstrated how Tchebycheff spaces can be characterized as finite-dimensional linear subspaces of $\mathscr{F}(T)$, whose elements are constrained to have a specified maximum number of alternations or zeros. Theorem 3.4 showed that a $(k+1)$-dimensional subspace of $\mathscr{F}(T)$ is a T-space if and only if for every $u \neq 0$, $S^+(u) \leqslant k$, $\max\{S^-(u), Z(u)\} \leqslant k$, or $S^0(u) + Z(u) \leqslant k$. These functions S^+, $\max\{S^-, Z\}$, $S^0 + Z$ as well as $S^0 + Z + D$ all therefore serve to indicate whether or not a finite-dimensional linear subspace in $\mathscr{F}(T)$ is a T-space. In fact, there are an infinite number of such functions. We call this family of functions indicator functions for T.

DEFINITION. A function $I: \mathscr{F}(T) \to \mathbb{Z}^+ \cup \{+\infty\}$ where $\mathbb{Z}^+$ is the set of nonnegative integers, is called an *indicator function* (for T) provided that for any $(k+1)$-dimensional subspace U of $\mathscr{F}(T)$, U is a T-space of degree k iff $I(u) \leqslant k$ for every nonzero $u \in U$.

Take any T-system $\{u_i\}_{i=0}^k$ $(k > 0)$ on a linearly ordered set T, card $T > k+1$, and any $t_0 \in T$. By changing the signs of $u_0(t_0),\dots, u_k(t_0)$ or, respectively, setting $u_0(t_0) = \cdots = u_k(t_0) = 0$, the respective linear spaces generated by the new u_i's are not T-spaces. However, the respective linear spaces are of dimension $k+1$, and for every $u \neq 0$ in the former, $Z(u) \leqslant k$ while for every $u \neq 0$ in the latter, $S^-(u) \leqslant k$, by application of Theorem 3.4. Thus, neither Z nor S^- are indicator functions. However, $S^-(u)$ and $Z(u)$ are both less than k for any nonzero element u belonging to any T-space of degree k.

We can introduce a partial ordering in the set of indicator functions for a set T as follows. If I_1, I_2 are any two, then $I_1 \leqslant I_2$ iff for every nonzero u, $I_1(u) \leqslant I_2(u)$.

We prove in this section that S^+ is the (unique) maximal element in the family of indicator functions for any subset of $\mathbb{R}$.

We now proceed to prove this. Actually, we prove a stronger result, namely that if $S^+(u)$ is finite then there is a Markov space of degree k containing u. This is constructed explicitly.

Before we proceed to the general proof we show how the proof works when u is a polynomial and T is some closed interval $[a, b]$. Let $S^+(u) = k$, and let all the zeros of u be simple in $[a, b]$. Then u has k distinct zeros in $[a, b]$, say $s_1 < \cdots < s_k$. We show that irrespective of the degree of u (as a polynomial), u can first be embedded into a T-space of degree k.

Let $P(t) = \prod_{i=1}^k (s_i - t)$. We assume for simplicity that $s_1 > a$ and $P(a) \cdot u(a) > 0$. Observe then that the polynomial $u(t)/P(t) > 0$ for all $t \in [a, b]$. Define the polynomials $u_i(t) = u(t)/(s_i - t)$ $(1 \leqslant i \leqslant k)$. We show that $u(t)$ together with $u_i(t)$ $(1 \leqslant i \leqslant k)$ form a T-ssytem of degree k in $\mathscr{F}([a, b])$ (and hence the space which they span is a T-space of degree k, containing of course $u(t)$).

Let $c_0, \dots, c_k$ be scalars, not all zero. It suffices to show that for $v(t) = c_0 u(t) + \sum_{i>0} c_i u_i(t)$, $Z(v) \leqslant k$ (recall that Z is an indicator function for continuous functions on closed intervals), since then also $u, u_1, \dots, u_k$ must be linearly independent. Define the polynomials $P_i(t) = P(t)/(s_i - t)$. Then

$v(t) = u(t)(c_0 - \sum c_i/(s_i - t)) = [u(t)/P(t)](c_0P(t) - \sum c_iP_i(t))$ is a polynomial all of whose zeros in $[a, b]$ are zeros of the polynomial $c_0P - \sum c_iP_i$ of degree $< k$. Hence $Z(v) < k$.

In the general case, it must be shown that $S^-(v) < k$, from which the desired result follows by Theorem 3.4. Difficulties arise because in general $u(t)/(s_i - t)$ is not well defined at $t = s_i$. This is handled by "splitting" the set T at each s_i.

Given that $S^+(u) < +\infty$ implies that u can be embedded into a T-space, it is shown from Lemma 4.1 that u can be embedded nito a Markov space.

(4.1) LEMMA. *Suppose $T \subset \mathbb{R}$, card $T \geq k$ and $x_1, x_2, \ldots, x_k \in \mathbb{R}$ satisfy $x_1 < x_2 < \cdots < x_k < \inf T$. Then any T-space of degree k on $\{x_1, x_2, \ldots, x_k\} \cup T$ is a Markov space of degree k on T.*

Proof. Let U be a T-space of degree k on $\{x_1, x_2, \ldots, x_k\} \cup T$, let $U_k = U$, and for $0 \leq i < k$ define U_i recursively by $U_i = \{u \in U_{i+1} : u(x_{i+1}) = 0\}$. Clearly, $U_0 \subset U_1 \subset \cdots \subset U_k = U$ and the restriction $U_i|_T$ is an $(i+1)$-dimensional subspace of $U|_T$. Furthermore, $U_i|_T$ is a T-space of degree i on T by the implication (3) $\Rightarrow$ (1) of Theorem 3.4.

Now suppose $T \subset \mathbb{R}$, $u \in \mathcal{F}(T)$ and $S^+(u) < +\infty$. Let θ be any strictly monotone bounded map, $\theta: \mathbb{R} \to \mathbb{R}$, set $\tilde{T} = \theta(T)$, and set $\tilde{v} = u \circ \theta^{-1} \in \mathcal{F}(\tilde{T})$. Then $S^+(u) = S^+(\tilde{v})$ and $\tilde{T}$ is bounded. Augment $\tilde{T}$ by $k = S^+(\tilde{v})$ points as in Lemma 4.1 and extend $\tilde{v}$ to v on the augmented set so that $S^+(v) = S^+(\tilde{v})$. It follows that if v is contained in a T-space of degree $S^+(v)$ on the augmented set, then $\tilde{v}$ is contained in a Markov space of degree $S^+(\tilde{v})$ on $\tilde{T}$ by Lemma 4.1. Hence, u is contained in a Markov space of degree $S^+(u)$ on $T = \theta^{-1}(\tilde{T})$.

(4.2) LEMMA. *Suppose T is linearly ordered, $u \in \mathcal{F}(T)$, $S^-(u) = k$, and $t \in T$. Then there exists a weak alternation sequence of length k for u, which includes t.*

Proof. Let $t_0 < \cdots < t_k$ be any weak alternation sequence for u of length k, and suppose (for example) that $t_n < t < t_{n+1}$. If $u(t) = 0$ then $t_0 < \cdots < t_n < t < t_{n+2} < \cdots < t_k$ is a sequence of the desired type. If $u(t) \neq 0$, find m $(0 \leq m \leq k)$ such that $u(t_m) \neq 0$ (possible since $Z(u) \leq S^-(u) = k$). Then either $(-1)^{n-m} u(t) u(t_m) \geq 0$ or $(-1)^{n+1-m} u(t) u(t_m) \geq 0$. Trading t_n for t in the first case, and t_{n+1} for t in the second, produces a sequence of the desired type.

(4.3) LEMMA. *Suppose P, $v \in \mathcal{F}(T)$ are such that $P(t)v(t) \geq 0$ and if $v(z) = 0$ then $P(z) = 0$. Then $S^+(v) \leq S^-(P)$.*

Proof. Suppose $t_0 < \cdots < t_k$ is a weak alternation sequence for v. Then $[(-1)^{i-j} v(t_i) v(t_j)][(-1)^{i-j} P(t_i) P(t_j)] = [v(t_i) P(t_i)][v(t_j) P(t_j)] \geq 0$. Since

$(-1)^{i-j} v(t_i) v(t_j) \geqslant 0$ and is equal to 0 only if $(-1)^{i-j} P(t_i) P(t_j) = 0$, it follows that $(-1)^{i-j} P(t_i) P(t_j) \geqslant 0$ whence $t_0 < \cdots < t_k$ is a weak alternation sequence for P. Thus $S^+(v) \leqslant S^+(P)$.

(4.4) THEOREM. *For an arbitrary $T \subset \mathbb{R}$ let $u \in \mathscr{F}(T)$ and suppose $S^+(u)$ is finite. Then there exists a Markov space of degree $S^+(u)$ containing u.*

Proof. In view of Lemma 4.1 and the remark following, it suffices to show that there is a T-space of degree $S^+(u)$ containing u. If $S^+(u) = 0$ then the one-dimensional space spanned by u is a T-space of degree 0. It is a similar triviality if $u = 0$. Hence, assume $u \neq 0$ and $S^+(u) = k > 0$. It follows that card $T \geqslant k + 1$. For any $x \in \mathbb{R}$ define $u_x = u|_{]-\infty,x]\cap T}$ (the restriction of u to $]-\infty, x] \cap T$). Note that when $]-\infty, x] \cap T \neq \varnothing$,

$$S^+(u_x) \leqslant S^+(u_y) \Leftarrow x \leqslant y, \tag{4.4.1}$$

and if the first inequality is strict, so must be the second.

Next it is shown that for $1 \leqslant i \leqslant k$, there exists an $x \in T$ such that

$$S^+(u_x) = i. \tag{4.4.2}$$

For $i = k$, since $S^+(u) = k$ there is a weak alternation sequence for u of length k, say $t_0 < \cdots < t_k$; then $x = t_k$ satisfies $S^+(u_x) = k$. Now suppose that y has been found such that $S^+(u_y) = i > 1$. We find an $x < y$ such that $S^+(u_x) = i - 1$ and the desired result then follows by reverse induction on i. Indeed, let $t_0 < \cdots < t_i \leqslant y$ be a weak alternation sequence for u_y. Then $t_0 < t_1 < \cdots < t_{i-1}$ is a weak alternation sequence for $u_{t_{i-1}}$, so $i - 1 \leqslant S^+(u_{t_{i-1}}) \leqslant S^+(u_y) = i$. If $S^+(u_{t_{i-1}}) = i - 1$ we are done, so assume $S^+(u_{t_{i-1}}) = i$ and let $s_0 < \cdots < s_i$ be a weak alternation sequence for $u_{t_{i-1}}$. Similarly, $i - 1 \leqslant S^+(u_{s_{i-1}}) \leqslant i$ and we are done unless $S^+(u_{s_{i-1}}) = i$ in which case we once more find a weak alternation sequence $r_0 < \cdots < r_i$ for $u_{s_{i-1}}$. But $r_i \leqslant s_{i-1} < s_i \leqslant t_{i-1} < t_i \leqslant y$ and $u(t_{i-1}) u(t_i) \leqslant 0$, so either $r_0 < \cdots < r_i < t_{i-1}$ or $r_0 < \cdots < r_i < t_i$ is a weak alternation sequence for u_y of length $i + 1 > S^+(u_y)$, a contradiction. Hence, either $S^+(u_{t_{i-1}}) = i - 1$ or else $S^+(u_{s_{i-1}}) = i - 1$, which completes the proof that for $1 \leqslant i \leqslant k$, there exists $x \in T$ such that (4.4.2) holds.

Define

$$s_i = \inf\{x \in \mathbb{R} \mid S^+(u_x) \geqslant i\}$$

for $1 \leqslant i \leqslant k$. By (4.4.2), $i - 1 \leqslant S^+(u_{s_i}) \leqslant i$ $(1 \leqslant i \leqslant k)$. It follows that $S^+(u_{s_i}) \leqslant i < i + 1 \leqslant S^+(u_{s_{i+2}})$, so

$$s_i < s_{i+2}. \tag{4.4.3}$$

Next we show that if $x, y \in T$, $x < y$ and $u(x) \neq 0$ then

$$S^+(u_x) + S^+(u|_{[x,y]\cap T}) = S^+(u_y). \tag{4.4.4}$$

Let $v = u|_{[x,y]\cap T}$. By Lemma 4.2 find a weak alternation sequence of length $S^+(u_x)$ for u_x including x and one of length $S^+(v)$ for v, also including x. Since $u(x) \neq 0$, the concatenation of the two sequences forms a weak alternation sequence for u_y, whence $S^+(u_x) + S^+(v) \leq S^+(u_y)$. On the other hand, there is a weak alternation sequence of length $S^+(u_y)$ for u_y, containing x, also by Lemma 4.2. This induces weak alternation sequences for u_x and v, so $S^+(u_x) + S^+(v) \geq S^+(u_y)$, completing the proof of (4.4.4).

It follows from the preceding that for any $x, y \in T$,

$$(-1)^{S^+(u_x) - S^+(u_y)}\, u(x)\, u(y) \geq 0. \tag{4.4.5}$$

Indeed, if $u(x)\,u(y) = 0$ then (4.4.5) is trivial, so assume $u(x)\,u(y) \neq 0$. Assume $x < y$, let $n = S^+(u_y)$ and find by Lemma 4.2 a weak alternation sequence of length n for u_y, including x, say $t_0 < \cdots < t_m = x < \cdots < t_n$. Then $m \leq S^+(u_x)$, $n - m \leq S^+(u|_{[x,y]\cap T})$ and thus by (4.4.4) $0 \leq [S^+(u_x) - m] + [S^+(u|_{[x,y]\cap T}) - (n - m)] = S^+(u_y) - n = 0$, so in particular $m = S^+(u_x)$. Furthermore, $(-1)^{m-n}\, u(x)\, u(t_n) > 0$ so if $(-1)^{m-n+1}\, u(x)\, u(y) > 0$ then $t_0 < \cdots < t_n < y$ is a weak alternation sequence for u_y of length $n+1$, an impossibility. Hence $(-1)^{m-n}\, u(x)\, u(y) \geq 0$, completing the proof of (4.4.5).

Define $s_0 = -\infty$, $s_{k+1} = +\infty$. Notice that if $s_i \leq x < s_{i+1}$ then $S^+(u_x) \geq i$ by the definition of s_i, whereas $S^+(u_x) < i + 1$ by the definition of s_{i+1}. Hence,

$$s_i \leq x < s_{i+1} \Rightarrow S^+(u_x) = i \quad (0 \leq i \leq k). \tag{4.4.6}$$

Furthermore, we obtain for $1 \leq i \leq k$

$$t \in \,]s_i, s_{i+1}[\, \cap T \Rightarrow u(t) \neq 0 \tag{4.4.7}$$

since if $u(t) = 0$ then $S^+(u_w) = S^+(u_t)$ whenever $s_i \leq w \leq t$; in this case, $]s_i, t[\, \cap T = \emptyset$ by (4.4.6), whence $i > S^+(u_{s_i}) = S^+(u_t) = i$, a contradiction.

Now since card $T > k$, there is a $t_* \in T$ different from $s_1, \ldots, s_k$, say $s_p < t_* < s_{p+1}$ $(0 \leq p \leq k)$. By (4.4.7) $u(t_*) \neq 0$, and we may, without loss of generality assume that $(-1)^p\, u(t_*) > 0$. With this normalization, since $S^+(u_{t_*}) = p$ by (4.4.6), for any $t \in T$ $(-1)^{S^+(u_t)}\, u(t) \geq 0$ by (4.4.5). Hence, by (4.4.6) and (4.4.7), we obtain

$$t \in \,]s_i, s_{i+1}[\, \cap T \Rightarrow (-1)^i\, u(t) > 0 \quad (0 \leq i \leq k). \tag{4.4.8}$$

The two results (4.4.3) and (4.4.8) were the goal of this first part of the proof of Theorem 4.4.

Next, a somewhat involved process is undertaken whose purpose is to "split" T at each "alternation" point s_i and to insert into the "split" a new "alternation" point r_i. This new point is isolated with respect to T, ensuring that $\varphi_i(t) = 1/(r_i - t)$ be well- defined (and bounded) in T. However, it is needed that the r_i's be alternation points of the new split set θT. To ensure this, when $s_i \in T$, s_i must fall on the appropriate side of r_i, according to the sign of $u(s_i)$: that is, on the left ($]s_{i-1}, s_i[$) side when $(-1)^{i-1} u(s_i) > 0$ and on the right ($]s_i, s_{i+1}[$) side when $(-1)^i u(s_i) > 0$. Define $\theta: \mathbb{R} \to \mathbb{R}$ by $\theta(t) = t + 4 \max\{i \mid s_i < t\}$ when $t \neq s_1, \ldots, s_k$; for $1 \leqslant i \leqslant k$, define $\theta(s_i) = s_i + 4i - 2$ except when $s_i = s_{i+1}$, in which case define $\theta(s_i) = s_i + 4i + 2$ $(= \theta(s_{i+1}))$. Clearly θ is strictly monotone on $\mathbb{R}$, as is θ^{-1} on the image of θ.

For $1 \leqslant i \leqslant k$, define

$$\begin{aligned} r_i &= \theta(s_i) - 1 && \text{if } s_i \in T \text{ and } (-1)^{i-1} u(s_i) > 0; \\ &= \theta(s_i) - 1 && \text{if } s_i \in T \text{ and } (-1)^i u(s_i) > 0, \text{ or } s_i = s_{i+1}; \\ &= \theta(s_i) && \text{if } s_i \notin T \text{ or } s_i \in T,\ u(s_i) = 0 \text{ and } s_i < s_{i+1}. \end{aligned}$$

In order to show that r_i is well defined, it is sufficient to show that if $s_i = s_{i+1}$ then $s_i \in T$ and $(-1)^i u(s_i) \geqslant 0$. Indeed, for $x < s_i$, $S^+(u_x) < i$ while for $x > s_i = s_{i+1}$, $S^+(u_x) > i$ (by the definition of s_i and s_{i+1}). Thus $x = s_i$ is the only element which can satisfy (4.4.2), whence $s_i \in T$ and $S^+(u_{s_i}) = i$. Furthermore, by the definition of s_i, since $s_i = s_{i+1}$, s_i must be an accumulation point of T from the right. Thus there is a $t \in]s_{i+1}, s_{i+2}[\cap T$, and by (4.4.6) $S^+(u_t) = i + 1$. But $(-1)^{i+1} u(t) > 0$ by (4.4.8) and $[(-1)^i u(s_i)] \times [(-1)^{i+1} u(t)] = (-1)^{i-(i+1)} u(s_i) u(t) \geqslant 0$ by (4.4.5), so $(-1)^i u(s_i) \geqslant 0$.

By construction, $r_1 < \cdots < r_k$, no r_i is an accumulation point of θT and $r_i \in \theta T$ only if $u(s_i) = 0$ and $s_i < s_{i+1}$, in which case $r_i = \theta(s_i)$. Since $(-1)^i u(t) > 0$ for $t \in]s_i, s_{i+1}[\cap T$, it follows from the above construction that whenever $t \in [r_i, r_{i+1}[\cap \theta T$, $(-1)^i u(\theta^{-1}(t)) \geqslant 0$ $(0 \leqslant i \leqslant k)$; equality occurs, of course, only when $t = r_i$.

A T-system defined on the set θT and including the function $u \circ \theta^{-1}$ is constructed. This T-system then pulls back to a T-system on T which includes u. Let $\varphi_0(t) = 1$ and let $\varphi_i(t) = 1/(r_i - t)$ for $1 \leqslant i \leqslant k$. For all $t \in \theta T$ and $0 \leqslant i \leqslant k$ define $u_i \in \mathscr{F}(\theta T)$ by $u_i(t) = u(\theta^{-1}(t))\, \varphi_i(t)$ unless $i > 0$ and $t = r_i$ in which case let $u_i(r_i) = (-1)^{i-1}$.

Notice that $u_0 = u \circ \theta^{-1}$. It will be demonstrated that $\{u_i\}_{i=0}^k$ is a T-system on θT. Assuming this is done, define $\hat{u}_i = u_i \circ \theta$ $(0 \leqslant i \leqslant k)$. Then $\hat{u}_0 = u$ and $\{\hat{u}_i\}_{i=0}^k$ is a T-system of degree k on T, which is equivalent to what was to be shown.

Hence, it suffices by Theorem 3.4 to show that the linear space generated by $\{u_i\}_{i=0}^k$ is of dimension $k + 1$ and that for any nonzero v therein, $S^+(v) \leqslant k$. For this it is sufficient to show that for any scalars $c_0, \ldots, c_k$, not all zero, if $v = \sum c_i u_i$ then $S^+(v) \leqslant k$ (since card $T > k$ and $Z(v) \leqslant S^+(v)$).

Define $P(t) = \prod_{i=1}^k (r_i - t)$. Then for all $t \in \theta T$, if $u(\theta^{-1}(t)) \neq 0$, then

$t \neq r_i$ $(1 \leqslant i \leqslant k)$ so $P(t) \neq 0$ and $u(\theta^{-1}(t))/P(t) > 0$ (since for $t \in\,]r_i, r_{i+1}[\cap \theta T$, both $(-1)^i u(\theta^{-1}(t)) > 0$ and $(-1)^i P(t) > 0$). Let $Q_i(t) = P(t)\,\varphi_i(t)$ $(0 \leqslant i \leqslant k)$ and set $Q(t) = \sum c_i Q_i(t)$. Then each Q_i and hence Q are all polynomials of degree less than or equal to k. Thus, by (3.2.2) $S^+(Q) \leqslant \deg Q \leqslant k$.

If $u(\theta^{-1}(t)) \neq 0$ then $v(t) = \sum c_i u_i(t) = [u(\theta^{-1}(t))/P(t)] \sum c_i P(t)\,\varphi_i(t) = [u(\theta^{-1}(t))/P(t)]\, Q(t)$. Since the term in brackets is strictly positive, $v(t)\,Q(t) = [u(\theta^{-1}(t))/P(t)]\, Q(t)^2 \geqslant 0$, and $v(t) = 0$ implies $Q(t) = 0$. On the other hand if $u(\theta^{-1}(t)) = 0$ then $t = r_j = \theta(s_j)$ for some $j = 1,\dots, k$ such that $s_j < s_{j+1}$. In this case $u_i(r_j) = u(s_j)\,\varphi_i(r_j) = 0$ when $i \neq j$ and $v(r_j) = \sum c_i u_i(r_j) = c_j(-1)^{j-1}$. At the same time $Q_i(r_j) = 0$ if $i \neq j$, so $Q(r_j) = \sum c_i Q_i(r_j) = c_j Q_j(r_j)$. Observe that $Q_j(t) = \prod_{i \neq j}(r_i - t)$, so $(-1)^{j-1} Q_j(r_j) > 0$. Hence $v(r_j)\,Q(r_j) = c_j^2(-1)^{j-1} Q_j(r_j) \geqslant 0$ and if $v(r_j) = 0$ then $c_j = 0$ so $Q(r_j) = 0$. Thus by Lemma 4.3, $S^+(v) \leqslant S^+(Q) \leqslant k$.

(4.5) Corollary. *For an arbitrary $T \subset \mathbb{R}$ let $u \in \mathscr{F}(T)$ and suppose $S^+(u) = k < +\infty$. If* card $T = n$ *then there exist T-spaces $U_i \subset \mathscr{F}(T)$ of respective degrees i, $i = 0, 1,\dots, n-1$ such that $u \in U_k$ and $U_0 \subset U_1 \subset \cdots \subset U_k \subset \cdots \subset U_{n-1}$. If* card T *is infinite then there exist T-spaces $U_i \subset \mathscr{F}(T)$ of respective degrees i, for all i, such that $u \in U_k$ and $U_0 \subset U_1 \subset \cdots \subset U_k \subset U_{k+1} \subset \cdots$.*

Proof. Shrink T to be a bounded set $\tilde{T}$ as in the discussion following Lemma 4.1. In the first case, find a finite set of points $r_{k+1}, r_{k+2},\dots, r_{n-1} \in \mathbb{R}$ satisfying $\sup \tilde{T} < r_{k+1} < r_{k+2} < \cdots < r_{n-1}$ and in the second case find a countably infinite set of points $r_{k+1}, r_{k+2},\dots \in \mathbb{R}$ satisfying $\sup \tilde{T} < r_{k+1} < r_{k+2} < \cdots$. Let $U_0 \subset U_1 \subset \cdots \subset U_k$ be the T-spaces U_i of respective degree i guaranteed by Theorem 4.4 with $u \in U_k$. For $i > k$ define $u_i(t) = u(t)/(r_i - t)$ on $\tilde{T}$ and let U_i be the $(i+1)$-dimensional subspace of $\mathscr{F}(\tilde{T})$ generated by U_k and $u_{k+1},\dots, u_i$. Then $U_k \subset U_{k+1} \subset U_{k+2} \subset \cdots$ and it can be shown, as in the proof of Theorem 4.4, that each U_i is a T-space of degree i, for $i > k$.

(4.6) Corollary. *Any indicator function I for a subset $T \subset \mathbb{R}$ satisfies $I(u) \leqslant S^+(u)$ for all nonzero $u \in \mathscr{F}(T)$.*

Proof. By Theorem 4.4, given $0 \neq u \in \mathscr{F}(T)$ there is a T-space of degree $S^+(u)$ containing u. Thus by definition, $I(u) \leqslant S^+(u)$.

Notes. (1) If u is bounded then by construction the elements of U_i are also bounded. It is unknown whether if u is continuous (respectively, n-differentiable) then there exists a T-space of degree $S^+(u)$ of continuous (respectively, n-differentiable) functions, containing u.

(2) It follows that when $T \subset \mathbb{R}$, $0 \neq u \in \mathscr{F}(T)$ is a member of some Markov (respectively, T-) space if and only if $S^+(u) < +\infty$.

Thus, another characterization of $S^+(u)$ derives, namely, $S^+(u)$ is the smallest degree of all T-spaces containing u.

(3) The obvious question, namely if U is an n-dimensional subspace of $\mathscr{F}(T)$ such that $S^+(u) < +\infty$ for each nonzero $u \in U$, does there exist a T-space containing U, provides an open problem which would be very worthwhile settling having, as it would, many applications in approximation theory. It appears difficult, however, even for the case $n = 2$.

The next theorem and corollary are used to show that no indicator function strictly dominates another in the sense that $I_1(u) < I_2(u)$ for all u such that $I_2(u) < +\infty$.

(4.7) THEOREM. *Let U be a $(k + 1)$-dimensional subspace of $\mathscr{F}(T)$, T linearly ordered. Given any indicator function I for T, there is a nonzero $u \in U$ such that $I(u) \geqslant k$.*

Proof. Suppose for every $0 \neq u \in U$, $I(u) < k$. Then every k-dimensional subspace of U is a T-space of degree $k - 1$ by definition of the indicator function. However, since U is $(k + 1)$-dimensional there exists by (1.1) some nonzero $v \in U$, such that v has at least k zeros. Consider some k dimensional subspace of U containing v. However, no linear space containing v is a T-space of degree $k - 1$ by Theorem 3.4 since v has k zeros. Therefore, there exists some nonzero element u in the subspace containing v such that $I(u) \geqslant k$.

(4.8) COROLLARY. *If U is a T-space of degree k then there is a nonzero $u \in U$ such that $I(u) = k$.*

(4.9) COROLLARY. *Let I_1, I_2 be any two indicator functions for the same set. Given any T-space U of degree k on this set there exist $u_1, u_2 \in U$ such that*

$$I_1(u_1) \leqslant I_2(u_1),$$
$$I_1(u_2) \geqslant I_2(u_2).$$

Proof. By Corollary 4.8 there exists $u_1 \in U$ such that $I_1(u_1) = k$ whence $I_2(u_1) \leqslant I_1(u_1) = k$; similarly, the required u_2 exists.

The Theorem 4.10 shows that no indicator function is subordinate to every indicator function.

(4.10) THEOREM. *Given an indicator function I for a set T of cardinality at least 2, there exists an indicator function J for T and a nonzero $u \in \mathscr{F}(T)$ such that $J(u) < I(u)$.*

Proof. Given T, let $U \subset \mathscr{F}(T)$ be a T-space of degree at least 1. By Theorem 4.7, $u \in U$ can be found such that $I(u) \geqslant 1$. Let J be defined on all real-valued functions by $J(v) = S^+(v)$ if $v \neq u$, $J(u) = 0$. It is easy to verify that J is an indicator function since any linear space containing u also contains αu for all real α, and for $\alpha \neq 0, 1$, $J(\alpha u) = S^+(\alpha u) = S^+(u)$.

While there is no minimal indicator function, suppose I is an indicator function which satisfies $I(\alpha u) = I(u)$ for all real numbers $\alpha \neq 0$, and which also satisfies $I(v) \leq I(u)$ whenever v is the restriction of u to a smaller domain; then $Z(u) \leq I(u)$ for all u. This is the content of the Theorem 4.11.

(4.11) Theorem. *Let I be any indicator function for a set T. Then for $0 \neq u \in \mathscr{F}(T)$ there is a nonzero $\alpha \in \mathbb{R}$ and a subset $S \subset T$ such that $Z(u) \leq I(\alpha u|_S)$.*

Proof. Let $S \subset T$ be the set of zeros of u together with some point t_0 such that $u(t_0) \neq 0$. We show that for some real α, $I(\alpha u|_S) \geq Z(u)$. Suppose $\max_{\alpha \neq 0} I(\alpha u|_S) = m < Z(u)$. Let $U \subset \mathscr{F}(S)$ be a T-space of degree m. Let V be the m-dimensional subspace of U such that for every $v \in V$, $v(t_0) = 0$. Consider the linear space W spanned by the elements of V and $u|_S$. Since $u(t_0) \neq 0$ dim $W = m + 1$. Any element $\omega \in W$ is of the form

$$\omega = v + a \cdot u|_S$$

where $v \in V$ and a is a scalar. Clearly $I(au|_S) \leq m$ for $a \neq 0$. We show that if $v \neq 0$ then $S^+(\omega) \leq m$ and hence $I(\omega) \leq m$ by Corollary 4.6. Indeed, $v(t) = \omega(t)$ when $t \neq t_0$, and $v(t_0) = 0$, so any generalized alternation sequence for ω is also one for v. Hence $S^+(\omega) \leq m$ if $v \neq 0$. Hence $I(\omega) \leq m$ for $\omega \neq 0$ belong to W, which implies that W is a T-space of degree m. However, u has more than m zeros and is contained in W, so W cannot be a T-space by Theorem 3.4, a contradiction. Therefore, $I(\alpha u|_S) \geq k$ for some α.

In Theorem 4.10 it was shown that there can be no minimal indicator function. In Theorem 4.11 it was shown that Z, while not an indicator function, does bound from below those indicator functions which satisfy a nominal normalizing condition. It is now shown that even among such "well-behaved" indicator functions, there can be no minimal element.

Indeed, we exhibit two "well-behaved" indicator functions N and M for which $\min\{N(\cdot), M(\cdot)\}$ is not an indicator function. It follows that there can be no indicator function subordinate to both N and M.

For any linearly ordered set T and any $u \in U \subset \mathscr{F}(T)$, let

$$M(u) = \max\{S^+(u), Z(u)\},$$

$$N(u) = \begin{cases} Z(u) & \text{if } S^0(u) = 0, \\ Z(u) + 1 & \text{if } S^0(u) \neq 0. \end{cases}$$

Then M is an indicator function by Theorem 3.4. We next show that N is an indicator function.

(4.12) LEMMA. *N is an indicator function.*

Proof. For any T-space U of degree k and any $0 \neq u \in U$, $N(u) \leqslant S^+(u) \leqslant k$ by Theorem 3.4. On the other hand, if U is a $(k+1)$-dimensional subspace of $\mathscr{F}(T)$ such that $N(u) \leqslant k$ for all nonzero $u \in U$, then $Z(u) \leqslant N(u) \leqslant k$ whenever $0 \neq u \in U$, so by Lemma 3.3, with respect to any basis for U, $\det V(t_0, ..., t_k) \neq 0$ whenever $t_0 < \cdots < t_k$ are elements of T. In order to show U is a T-space, by Theorem 3.4 it remains to show that the above determinant has permanence of sign. The proof of this is exactly the proof that (3) $\Rightarrow$ (1) in Theorem 3.4.

It remains to show that

$$\nu(u) = \min\{M(u), N(u)\}$$

is not an indicator function. To this end, we first exhibit a general method for constructing Haar spaces which are not T-spaces.

Indeed, let $U \subset \mathscr{F}(T)$ be any T-space of degree k, where card $T > 2$, and let $t_* \in T$ satisfy $\inf T < t_* < \sup T$. For each $u \in U$ define $u_* \in \mathscr{F}(T)$ by

$$\begin{aligned} u_*(t) &= u(t) \qquad \text{if} \quad t \leqslant t_*, \\ &= -u(t) \qquad \text{if} \quad t > t_*, \end{aligned}$$

Set $U_* = \{u_* \in \mathscr{F}(T) \mid u \in U\}$. With these definitions, we obtain the following.

(4.13) LEMMA. *Given any T-space $U \subset \mathscr{F}(T)$ where* card $T > 2$, *then for any choice of t_*, U_* is a Haar space but is not a T-space.*

Proof. Suppose the dimension of U is $k+1$. Then the dimension of U_* is also $k+1$. Furthermore, for each nonzero $u_* \in U_*$, $Z(u_*) = Z(u) \leqslant k$ by Theorem 3.4. Thus U is a Haar space.

Let $\varphi \in U$ have k distinct zeros, one of which is t_* (cf. (2.2)). By Lemma 3.1 $D(\varphi) = 0$, so in particular t_* is not a double zero of φ. Thus, t_* is a double zero of φ_* and $D(\varphi_*) = 1$. Hence, again by Lemma 3.1, $S^+(\varphi_*) > k$ so U_* is not a T-space by Theorem 3.4.

Now, let T satisfy $2 <$ card T, and let $U \subset \mathscr{F}(T)$ be any T-space such that the degree k of U satisfies $1 + 2k \geqslant$ card T. Then, for any nonzero $u_* \in U_*$, $\nu(u_*) \leqslant N(u_*) \leqslant Z(u_*) + 1$ so if $Z(u_*) < k$ then $\nu(u_*) \leqslant k$. On the other hand, if $Z(u_*) = k$ then $S^-(u_*) \leqslant k$ whence $\nu(u_*) \leqslant M(u_*) = k$.

Thus, for each nonzero $u_* \in U_*$, $\nu(u_*) \leqslant k$. However, U_* is not a T-space by Lemma 4.13, and thus ν is not an indicator function.

From the embedding Theorem 4.4 for T-spaces, follows an analogous result for Haar spaces.

(4.14) COROLLARY. *For an arbitrary $T \subset \mathbb{R}$ let $u \in \mathscr{F}(T)$ and suppose $Z(u)$ is finite. Then there exists a Haar space of degree $Z(u)$ containing u.*

Proof. It is easy to find $\phi \in \mathscr{F}(T)$ such that $Z(\phi) = 0$ and $S^-(\phi \cdot u) = Z(u)$. If U is the T-space containing $\phi \cdot u$ guaranteed by Theorem 4.4, then $\{v/\phi \mid v \in U\}$ is clearly the desired Haar space.

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